Master in Actuarial Science

Risk Theory

31-05-2011

Solution

1. -

   (a) $\Pr\{W_2 < 5\} = F_{W_2}(5) = \Pr\{N(5) \geq 2\} = 1 - \Pr\{N(5) \leq 1\} = 1 - \exp(-2.5) - \exp(-2.5) * 2.5 = 0.712\,7$

   (b) $\Pr\{5 < W_2 < 10\} = F_{W_2}(10) - F_{W_2}(5) = \Pr\{N(10) \geq 2\} - \Pr\{N(5) \geq 2\} = \Pr\{N(5) \leq 1\} - \Pr\{N(10) \leq 1\} =$
   $= \exp(-2.5) + \exp(-2.5) * 2.5 - \exp(-5) - \exp(-5) * 5 = 0.246\,87$

   (c) $E[X] = \frac{\theta}{\alpha - 1} = 1$ and $V[X] = \frac{\alpha\theta^2}{(\alpha-2)(\alpha-1)^2} = 3$. Then

   $$\begin{cases} \frac{\theta}{\alpha-1} = 1 \\ \frac{\alpha}{\alpha-2} = 3 \end{cases}$$

   which is to say $\alpha = 3$ and $\theta = 2$. Hence $\Pr\{X > 8\} = \left(\frac{2}{2+8}\right)^3 = 0.008$. Hence the number of claims reported to the reinsurer follows a Poisson Process with rate $0.5 * 0.008 = 0.004$

   This implies that S_1 is an exponential random variable with mean $1/0.004 = 250$ (very rarely is reported a claim to the reinsurer). Then $\Pr\{S_1 \leq 10\} = 1 - \exp(-10/250) = 0.03\,921\,1$.

2. -

   (a)

   $$M = \sum_{i=0}^{N} I_i$$

where $\{I_i\}_{i=1,2,...}$ are i.i.d. random variable, Bernoulli distributed, with parameter $p = \Pr\{X > d\} = 1 - F_X(d)$. Then

$$\begin{aligned} P_M(z) &= E\left[z^{\sum_{i=0}^{N} I_i}\right] = E\left[E\left[z^{\sum_{i=0}^{N} I_i}|N\right]\right] = \\ &= E\left[\left(E\left[z^I\right]\right)^N\right] = E\left[(zp + (1-p))^N\right] = \\ &= P_N(p(z-1)+1) = (1 - \beta p(z-1))^{-r}, \end{aligned}$$

which is the probability generating function of a negative binomial with parameters r and $\beta p = \beta(1 - F_X(d))$.

(b) Attending to the fact that the Gamma parameter α is integer (2) we can use the Poisson distribution to calculate $1 - F_X(2) = \Pr\{X > 2\} = \exp(-2) + \exp(-2) * 2 = 0.406\,01$ (otherwise use the χ^2 table, or integrate by parts the density).

Hence M is a negative binomial with parameters 2.5 and 0.406 01. Hence $\Pr\{M = 2\} = \frac{3.5*2.5*0.406\,01^2}{2*(1+0.406\,01)^{4.5}} = 0.155\,63$.

3. -

(a)

$$p_k(t) = \int_0^\infty \mathrm{e}^{-\lambda t} \frac{(\lambda t)^k}{k!} dU(\lambda), \; k = 0, 1, 2, ...$$

(b) Given $\Lambda = \lambda$ the process is a Poisson process, which has independent increments. Hence

$$\begin{aligned} p_{k,k+n}(s,t) &= \Pr\{N(t) - N(s) = n | N(s) = k\} = \\ &= \frac{\Pr\{N(t) - N(s) = n, N(s) = k\}}{\Pr\{N(s) = k\}} = \\ &= \frac{1}{p_k(s)} \int_0^\infty \Pr\{N(t) - N(s) = n, N(s) = k | \lambda\} dU(\lambda) = \\ &= \frac{1}{p_k(s)} \int_0^\infty \Pr\{N(t) - N(s) = n | \lambda\} \Pr\{N(s) = k | \lambda\} dU(\lambda) = \\ &= \frac{1}{p_k(s)} \int_0^\infty \frac{e^{-\lambda(t-s)} \left[\lambda(t-s)\right]^n}{n!} \frac{e^{-\lambda s} (\lambda s)^k}{k!} dU(\lambda) = \\ &= \frac{1}{p_k(s)} \frac{(t-s)^n}{n!k!} s^k \int_0^\infty e^{-\lambda t} \lambda^{n+k} dU(\lambda) = \\ &= \frac{1}{p_k(s)} \frac{(t-s)^n}{n!k!} \frac{s^k (n+k)!}{t^{n+k}} \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^{n+k}}{(n+k)!} dU(\lambda) \\ &= \binom{k+n}{n} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^n \frac{p_{k+n}(t)}{p_k(s)}. \end{aligned}$$

4.

$$
\begin{aligned}
f_S(x) &= \frac{\frac{\beta}{1+\beta}\sum_{y=1}^{\min(x,4)} f_X(y) f_S(x-y)}{1-\frac{\beta}{1+\beta} f_X(0)} \\
f_S(0) &= [1-\beta(f_X(0)-1)]^{-1}
\end{aligned}
$$

$f_X(0) = f_X(1) = f_X(2) = f_X(3) = f_X(4) = 1/5$; $\beta = 5$; $\frac{\beta}{1+\beta} = 5/6$. Then

$$f_S(0) = 1/5$$

$$f_S(x) = \frac{\sum_{y=1}^{x} f_S(x-y)}{5},$$

from where we get

$$
\begin{aligned}
f_S(1) &= \frac{1}{25} \\
f_S(2) &= \frac{6}{125} \\
f_S(3) &= \frac{36}{625} \\
F_S(3) &= \frac{216}{625}.
\end{aligned}
$$

5. $VaR_{0.99}(X) = \theta(0.01^{-0.5} - 1) = 900 \Rightarrow \theta = 100$

$$E(X \wedge 50) = 100(1 - 100/150) = 100/3.$$

6. -

   (a) $P = \int_0^\infty (S_X(x))^{1/\rho}\, dx = \int_0^\infty \left(\frac{\theta}{\theta+x}\right)^{\alpha/\rho} dx = \frac{\theta}{\alpha/\rho - 1}$

   (b) $P_{re} = \int_M^\infty \left(\frac{\theta}{\theta+x}\right)^{\alpha/\rho} dx = \frac{\theta^{\alpha/\rho}}{\alpha/\rho-1}(\theta+M)^{1-\alpha/\rho}$ (it could be calculated as $E[Y] - E[Y \wedge M]$, where Y is a Pareto with parameters $(\alpha/\rho, \theta)$.

7. -

   (a) X is a Gamma with parameters $\alpha = 2$ and $\theta = 5/3$. Hence $E[X] = 10/3$ and $E[X^2] = \frac{\left(\frac{5}{3}\right)^2 \Gamma(4)}{\Gamma(2)} = \left(\frac{5}{3}\right)^2 * 6 = \frac{50}{3}$. Then the premium is $c = \lambda\left(10/3 + 0.1 * \frac{50}{3}\right) = 5\lambda$.

   (b) From a) we have that the loading coefficient on the expected value principle is 0.5. hence $\psi(0) = \frac{1}{1.5} = 0.666\,67$.

(c) The adjustment coefficient is the only positive root of

$$\begin{aligned} 5r &= M_X(r) - 1 \\ &\Leftrightarrow \\ 5r &= (1 - 5/3r)^{-2} - 1 \end{aligned}$$

The only positive root of this equation, satisfying $r < 1/\theta = 3/5$, is $R = \frac{1}{2} - \frac{1}{10}\sqrt{13} = 0.139\,44$.

8.

$$\begin{aligned} E[X] &= \exp(2 + 0.5) = 12.\,182 \\ E[X^2] &= \exp(4 + 2) = 403.\,428 \\ E[X^3] &= \exp(6 + 4.5) = 36315.503 \end{aligned}$$

$$\begin{aligned} \mu_S &= 1000 * 12.\,1825 = 12182 \\ \sigma_S &= 403428^{0.5} = 635.16 \\ \gamma_S &= 36315503/403428^{1.5} = 0.141\,72 \end{aligned}$$

$$\begin{aligned} \Pr\left\{\frac{S}{P} < 1.05\right\} &\simeq 0.99 \\ &\Leftrightarrow \\ \Pr\left\{\frac{S - \mu_S}{\sigma_S} < \frac{1.05P - 12182}{635.16}\right\} &\simeq 0.99 \end{aligned}$$

Using, for instance, the NP approximation we get, $z_{0.01} = 2.3263$, from where

$$z_{0.01} + \frac{0.141\,72}{6}(z_{0.01}^2 - 1) = 2.3263 + \frac{0.141\,72}{6}(2.3263^2 - 1) = 2.\,430\,5$$

which is to say $\frac{1.05P - 12182}{635.16} = 2.\,430\,5 \Leftrightarrow P = 13072.$